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NEW ANALYTICAL METHOD FOR A NONLINEAR DIFFUSION PROBLEM

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Abstract—Nonlinear parabolic partial differential equation with a concentration dependent diffusivity in semi-infinite region is solved by an analytical method under a constant flux boundary condition. The differential equation is transformed to a system of simultaneous linear ordinary differential equations. The solutions are represented by a series of products of the repeated integrals of the error function.

NOMENCLATURE

A, coefficient;
a, coefficient;
B, coefficient;
b, thickness of sample [m];
$\mathbf{D}$, relative diffusivity, $= D(\phi)/D(\phi_i)$;
D, diffusivity [m^2/h];
$\mathbf{E}$, repeated integral of the complemental error function;
E, relative change of diffusivity, $= D(\phi_i)/D(\phi = 0)$;
F, function;
$\mathbf{N}_i$, relative intensity of flux, $= N_i b/\phi_i D(\phi_i)$;
N_i, mass flux at surface [kg/m^2 h];
T, dimensionless time, $= D(\phi_i)t/b^2$;
t, time [h];
U, dimensionless concentration (equation 4);
U_s, dimensionless concentration due to Storm (equation 42);
V, dimensionless concentration (equation 17);
X, dimensionless length, $= x/b$;
x, length from surface [m];
Z, dimensionless length (equation 17);
z, dimensionless length (equation 42).

Greek symbols

α, exponent;
β, exponent;
ζ, dimensionless concentration (equation 42);
η, similarity variable, $= X/2\sqrt{T}$;
θ, auxiliary variable;
κ, parameter for relative diffusivity;
ν, auxiliary variable;
ξ, auxiliary variable;
τ, dimensionless time, $= \kappa^2 T$;
ϕ, concentration [kg/m^3];
ϕ_i, initial concentration [kg/m^3];
χ, auxiliary variable.

INTRODUCTION

THERE are some reasons to suppose that the moisture movement within a wet porous material during drying process may, under certain conditions, be described in terms of the diffusion equation [1, 2], in which the apparent diffusivity, however, is usually a function of the moisture concentration. The unsteady state moisture distribution within the wet stock during the initial stage of drying is, thus, described by the following nonlinear diffusion equation;

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left\{D\frac{\partial \phi}{\partial x}\right\} \text{ in } 0 \leqslant x \leqslant b,\ t > 0. \tag{1}$$

The initial and the boundary conditions for this equation are described by;

$$\left.\begin{array}{lll} \phi = \phi_i & \text{in } 0 \leqslant x \leqslant b, & t = 0 \\ D\dfrac{\partial \phi}{\partial x} = N_i & \text{at } x = 0, & t > 0 \\ D\dfrac{\partial \phi}{\partial x} = 0 & \text{at } x = b, & t > 0 \end{array}\right\}. \tag{2}$$

Since the restriction of the finiteness in the coordinate x causes a great mathematical difficulty, the exact analytical solutions for such a nonlinear diffusion problem in the finite region are not known to date. Fortunately, it has been shown that if the flux N_i or the thickness b is sufficiently large, we can approximate the bounded region as semi-infinite, during the initial stage of the drying process [1, 3]. The system of the equations, then, reduces to the following nonlinear diffusion equation in the semi-infinite region.

$$\left.\begin{array}{lll} \dfrac{\partial \phi}{\partial t} = \dfrac{\partial}{\partial x}\left\{D\dfrac{\partial \phi}{\partial x}\right\} & \text{in } 0 \leqslant x < \infty, & t > 0 \\ \phi = \phi_i & \text{in } 0 \leqslant x < \infty, & t = 0 \\ D\dfrac{\partial \phi}{\partial x} = N_i & \text{at } x = 0, & t > 0 \end{array}\right\}. \tag{3}$$

The flux boundary condition in this system of equations, however, makes it still very difficult to solve by an analytical method. The only exact solution has been presented by Storm [4] for such a system. Though his analysis has been extended to some problems by Knight and Philip [5], their solutions are valid only when the diffusivity obeys a certain functional form of the concentration. Besides, most of their solutions are not described in an explicit form.

In this article, a general analytical solution is presented for the nonlinear diffusion problem in the semi-infinite region subject to the constant flux boundary condition.

QUASI-LINEARIZATION

The Kirchhoff transformation;

$$U = \frac{1}{N_i b}\int_{\phi}^{\phi_i} D(\phi)\,\mathrm{d}\phi \tag{4}$$

and the following dimensionless parameter and variables*

$$\left.\begin{aligned} T &= D(\phi_i)t/b^2 \\ X &= x/b \\ \mathbf{D} &= D(\phi)/D(\phi_i) \end{aligned}\right\}. \tag{5}$$

reduce the system of the equation (3) to the following nondimensional quasi-linear diffusion equations;

$$\left.\begin{aligned} &\frac{\partial U}{\partial T} = \mathbf{D}\frac{\partial^2 U}{\partial X^2} && \text{in } 0 \leqslant X < \infty, && T > 0 \\ &U = 0 && \text{in } 0 \leqslant X < \infty, && \text{at } T = 0 \\ &\frac{\partial U}{\partial X} = -1 && \text{at } X = 0, && T > 0 \end{aligned}\right\} \tag{6}$$

where, the dimensionless relative diffusivity $\mathbf{D}$ is regarded as a function of U. The functional form of $\mathbf{D}(U)$ depends directly on that of $D(\phi)$ and may be evaluated by making use of the inverse of the Kirchhoff transformation. If the apparent diffusivity can be approximated, for example, by the following exponential type equation;

$$D(\phi) = A\cdot\exp(B\cdot\phi) \tag{7}$$

the dimensionless diffusivity, then is described by

$$\mathbf{D}(U) = 1-\kappa\cdot U. \tag{8}$$

The parameter κ in this equation is defined by the following equation;

$$\kappa = \mathbf{N}_i \ln E \tag{9}$$

where

$$\mathbf{N}_i = N_i b/\phi_i D(\phi_i) \tag{10}$$

is the dimensionless flux density and

$$E = D(\phi_i)/D(\phi = 0) \tag{11}$$

is the relative change of the diffusivity. Some examples of the functional form of $\mathbf{D}$ and the definition of the parameter κ for various types of the diffusivity $D(\phi)$ are listed in Table 1 [6].

As seen from Table 1, the functional form of the relative diffusivity $\mathbf{D}$ may, in many cases, be described by the following series form;

$$\mathbf{D}(U) = 1-a_1\kappa U - a_2(\kappa U)^2 - a_3(\kappa U)^3 - \ldots \tag{12}$$

The coefficients a_j in this equation depend on the functional type of the diffusivity $D(\phi)$ but do not depend on the flux density N_i nor the relative change of the diffusivity E.

Table 1. Examples of the definitions of the relative diffusivities and the parameters

$D(\phi)$	$\mathbf{D}(U)$	κ
$A\cdot\exp(B\cdot\phi)$	$1-\kappa\cdot U$	$\mathbf{N}_i \ln E$
$(A\cdot\phi+B)^{-1}$	$\exp(-\kappa\cdot U)$	$\mathbf{N}_i(E-1)$
$(A\cdot\phi+B)^n$	$(1-\kappa\cdot U)^{n/(n+1)}$	$\mathbf{N}_i(n+1)(1-E^{-1/n})$

Substitution of equation (12) into equation (6) yields

$$\left.\begin{aligned} &\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} \\ &\qquad -\{a_1\kappa U + a_2(\kappa U)^2 + a_3(\kappa U)^3 + \ldots\}\frac{\partial^2 U}{\partial X^2} \\ &U = 0 \quad \text{at} \quad T = 0 \\ &\frac{\partial U}{\partial X} = -1 \quad \text{at} \quad X = 0. \end{aligned}\right\} \tag{13}$$

Applying the perturbation method, one can easily prove that the solution can be expressed by

$$U = U_1 + \kappa U_2 + \kappa^2 U_3 + \kappa^3 U_4 + \ldots \tag{14}$$

where, the unknown functions U_j are defined by the following equations;

$$\left.\begin{aligned} U_1 &= 2\sqrt{T}\left\{\frac{1}{\sqrt{\pi}}\exp\left(-\frac{X^2}{4T}\right) - \frac{X}{2\sqrt{T}}\operatorname{erfc}\left(\frac{X}{2\sqrt{T}}\right)\right\} \\ U_2 &= \mathscr{L}\left[a_1 U_1\frac{\partial^2 U_1}{\partial X^2}\right] \\ U_3 &= \mathscr{L}\left[a_1\left(U_1\frac{\partial^2 U_2}{\partial X^2} + U_2\frac{\partial^2 U_1}{\partial X^2}\right) + a_2 U_1^2\frac{\partial^2 U_1}{\partial X^2}\right] \\ U_4 &= \mathscr{L}\left[a_1\left(U_1\frac{\partial^2 U_3}{\partial X^2} + U_2\frac{\partial^2 U_2}{\partial X^2} + U_3\frac{\partial^2 U_1}{\partial X^2}\right) + a_2\left(U_1^2\frac{\partial^2 U_2}{\partial X^2} + 2U_1U_2\frac{\partial^2 U_1}{\partial X^2}\right) + a_3 U_1^3\frac{\partial^2 U_1}{\partial X^2}\right] \\ &\ldots \\ &\ldots\ldots \end{aligned}\right\} \tag{15}$$

The operator $\mathscr{L}$ in these equations is defined by the following integral transform;

$$\mathscr{L}[f] \equiv \frac{1}{2\sqrt{\pi}}\int_0^T \mathrm{d}\theta \int_0^{\infty} \times \frac{\exp\left\{-\dfrac{(X-\xi)^2}{4(T-\theta)}\right\} + \exp\left\{-\dfrac{(X+\xi)^2}{4(T-\theta)}\right\}}{(T-\theta)^{1/2}} \times f(\theta,\xi)\,\mathrm{d}\xi. \tag{16}$$

*Though the thickness of the sample b is used in these transformations, another characteristic length, $D(\phi)\rho/N_i$ or unit length for example, may be used instead of it.

If we can evaluate the integral transforms in equation (15) by an analytical or a numerical method, we can obtain the solution U by making use of equation (14), but the integration procedures are practically impossible. In the following section, another analytical approach to the general solution is discussed.

SIMILARITY ANALYSIS

Further transformations;

$$\left.\begin{aligned} V &= \kappa U \\ \tau &= \kappa^2 T \\ Z &= \kappa X \end{aligned}\right\} \tag{17}$$

reduce equation (13) to

$$\left.\begin{aligned} &\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial Z^2} - (a_1 V + a_2 V^2 + a_3 V^3 + \ldots)\frac{\partial^2 V}{\partial Z^2} \\ &V = 0 \quad \text{at} \quad \tau = 0 \\ &\frac{\partial V}{\partial Z} = -1 \quad \text{at} \quad Z = 0. \end{aligned}\right\} \tag{18}$$

The parameter κ in equation (13) is eliminated by the transform (17). Hence the solution $V(\tau, Z)$ for equation (18) should coincide with that of equation (13) for the case of $\kappa = 1$, one can readily write;

$$U(\tau, Z)|_{\kappa=1} = V(\tau, Z). \tag{19}$$

Applying the transforms (17), equation (19) becomes;

$$U(\tau, Z)|_{\kappa=1} = \kappa U(\tau/\kappa^2, Z/\kappa)|_{\kappa}. \tag{20}$$

Substitution of equation (14) into (20) yields the following relations;

$$\left.\begin{aligned} &U_1(\tau, Z) = \kappa U_1(\tau/\kappa^2, Z/\kappa) \\ &U_2(\tau, Z) = \kappa^2 U_2(\tau/\kappa^2, Z/\kappa) \\ &U_3(\tau, Z) = \kappa^3 U_3(\tau/\kappa^2, Z/\kappa) \\ &\ldots \\ &\ldots. \end{aligned}\right\} \tag{21}$$

One can easily find that the following general functional form satisfies equation (21),*

$$U_j(T, X) = T^{\alpha} \cdot X^{\beta} \cdot F_j(\eta), \; j = 1, 2, 3, \ldots \tag{22}$$

where, the exponents α and β satisfy

$$2\alpha + \beta = j \tag{23}$$

and F is an unknown function of arbitrally similarity variable η;

$$\eta = \eta(X^2/T). \tag{24}$$

We may use the simplest and well known definition;

$$\eta = X/2\sqrt{T}. \tag{25}$$

Hence the solution U is not always zero at the surface ($X = 0$), the exponent β is discarded from equation (22). We finally find that;

$$\left.\begin{aligned} &U_1(T, X) = T^{0.5} F_1(\eta) \\ &U_2(T, X) = T^{1.0} F_2(\eta) \\ &U_3(T, X) = T^{1.5} F_3(\eta) \\ &U_4(T, X) = T^{2.0} F_4(\eta) \\ &\ldots \\ &\ldots. \end{aligned}\right\} \tag{26}$$

Equations (14) and (26) yield;

$$U = T^{0.5} F_1(\eta) + \kappa T F_2(\eta) + \kappa^2 T^{1.5} F_3(\eta) + \ldots \tag{27}$$

and

$$\left.\begin{aligned} \frac{\partial U}{\partial T} &= \{(F_1 - \eta F_1')T^{-0.5} + \kappa(2F_2 - \eta F_2') \\ &\qquad + \kappa^2(3F_3 - \eta F_3')T^{0.5} + \ldots\}/2 \\ \frac{\partial U}{\partial X} &= \{F_1' + \kappa F_2' T^{0.5} + \kappa^2 F_3' T^{1.0} \\ &\qquad + \kappa^3 F_4' T^{1.5} + \ldots\}/2 \\ \frac{\partial^2 U}{\partial X^2} &= \{F_1'' T^{-0.5} + \kappa F_2'' + \kappa^2 F_3'' T^{0.5} \\ &\qquad + \kappa^3 F_4'' T^{1.0} + \ldots\}/4. \end{aligned}\right\} \tag{28}$$

Substituting equations (28), (27), (26) and (14) into (13), and collecting coefficients of like powers of the parameter κ, lead the following simultaneous ordinary differential equations for F_j;

$$\left.\begin{aligned} &F_1'' + 2\eta F_1' - 2F_1 = 0 \\ &F_2'' + 2\eta F_2' - 4F_2 = a_1 F_1 F_1'' \\ &F_3'' + 2\eta F_3' - 6F_3 = a_1(F_1 F_2'' + F_2 F_1'') \\ &\qquad + a_2 F_1^2 F_1'' \\ &F_4'' + 2\eta F_4' - 8F_4 = a_1(F_1 F_3'' + F_2 F_2'' + F_3 F_1'') \\ &\qquad + a_2(F_1^2 F_2'' + 2F_1 F_2 F_1'') \\ &\qquad + a_3 F_1^3 F_1'' \\ &\ldots \\ &\ldots. \end{aligned}\right\} \tag{29}$$

The conditions at $X = 0$;

$$\left.\frac{\partial U}{\partial X}\right|_{X=0} = -1 \tag{30}$$

and at $X = \infty$;

$$U(X \to \infty) = 0 \tag{31}$$

yield the following boundary conditions for F_j;

$$\left.\begin{aligned} F_1'(\eta = 0) &= -2 \\ F_2'(\eta = 0) &= F_3'(\eta = 0) \\ &= F_4'(\eta = 0) = \ldots = 0 \\ F_1(\eta \to \infty) &= F_2(\eta \to \infty) \\ &= F_3(\eta \to \infty) = \ldots = 0. \end{aligned}\right\} \tag{32}$$

This system of the two point boundary value problem for the simultaneous linear ordinary differential equations (29) can be easily solved by an analytical or a numerical method. The dimensionless concentration U, then, can be evaluated by making use of equations (26) and (14). The concentration distribution $\phi(T, X)$ can be calculated by the inverse of the Kirchhoff transformation (4). Some examples are discussed in the following section.

*Functions U_j are constant conformally invariant under the one parameter continuous transformation group (17) [9]. Therefore, U_j can be represented by a product of the absolute invariant F and the conformally invariant $T^{\alpha}X^{\beta}$.

EXAMPLES AND DISCUSSIONS

1. *Quadratical diffusion coefficient*

If the diffusion coefficient is inversely proportional to a quadratic equation of concentration;

$$D(\phi) = (A \cdot \phi + B)^{-2} \tag{33}$$

the Kirchhoff transformation

$$U(\phi) = \frac{1}{N_i b} \int_{\phi}^{\phi_i} D \, \mathrm{d}\phi = \frac{1 - \sqrt{\mathbf{D}}}{\mathbf{N}_i(\sqrt{E} - 1)} = \frac{1}{\kappa}(1 - \sqrt{\mathbf{D}}) \tag{34}$$

yields a quadratic equation for the relative diffusivity;

$$\left.\begin{aligned} \mathbf{D} &= (1 - \kappa U)^2 \\ \kappa &= \mathbf{N}_i(\sqrt{E} - 1). \end{aligned}\right\} \tag{35}$$

The coefficients a_j in equation (12) then become

$$\left.\begin{aligned} a_1 &= 2 \\ a_2 &= -1 \\ a_3 &= a_4 = \ldots = 0. \end{aligned}\right\} \tag{36}$$

The simultaneous ordinary differential equations (29) and (32) then reduce to the following equations.

$$\left.\begin{aligned} & F_1'' + 2\eta F_1' - 2F_1 = 0 \\ & F_1'(0) = -2 \\ & F_1(\infty) = 0 \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} & F_2'' + 2\eta F_2' - 4F_2 = 2F_1 F_1'' \\ & F_2'(0) = 0 \\ & F_2(\infty) = 0 \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} & F_3'' + 2\eta F_3' - 6F_3 = 2(F_1 F_2'' + F_2 F_1'') - F_1^2 F_1'' \\ & F_3'(0) = 0 \\ & F_3(\infty) = 0 \\ & \ldots \\ & \ldots . \end{aligned}\right\} \tag{39}$$

The analytical solutions for these equations are given by (Appendix);

$$\left.\begin{aligned} F_1 &= 2\mathbf{E}_1 \\ F_2 &= -6\mathbf{E}_2 + 4\mathbf{E}_2\mathbf{E}_0 \\ F_3 &= 16\mathbf{E}_3 - 12(\mathbf{E}_3\mathbf{E}_0 + \mathbf{E}_2\mathbf{E}_1) \\ &\quad + 4\mathbf{E}_2\mathbf{E}_2\mathbf{E}_{-1} + 8\mathbf{E}_2\mathbf{E}_1\mathbf{E}_0 \end{aligned}\right\} \tag{40}$$

where, $\mathbf{E}_n$ is defined by the repeated integral of the complemental error functions as;

$$\mathbf{E}_n(\eta) = i^n \cdot \mathrm{erfc}(\eta). \tag{41}$$

Further analytical solutions F_4, $F_5, \ldots$ are not known, but the ordinary differential equation (36) can be solved easily by numerical method such as the Runge–Kutta method. Some examples of the numerical results are shown in Fig. 1. The numerical values of these functions F_j at the surface ($\eta = 0$) are listed in the Table 2. We can evaluate the unsteady state concentration profile $U(T, X)$ and the transient change of U at the surface by making use of these numerical values and equation (27). The calculated results are shown in Figs. 2 and 3.

A strict analytical solution has been presented by Storm [4] for this special case of the diffusivity. The

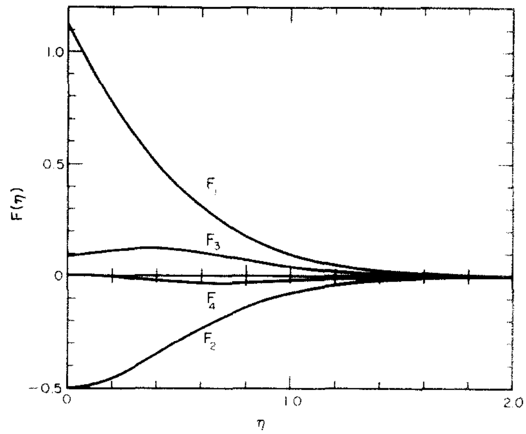

FIG. 1. Examples of the functions F_j for the case of $\mathbf{D}(U) = (1 - \kappa U)^2$.

Table 2. Values of $F_j(0)$ for $D = (A\phi + B)^{-2}$

$F_1(0)$	1.128379167	$(= 2/\sqrt{\pi})$
$F_2(0)$	-0.5	$(= -1/2)$
$F_3(0)$	0.094031597	$(= 1/6\sqrt{\pi})$
$F_4(0)$	0.0	
$F_5(0)$	-0.002350789	$(= -1/240\sqrt{\pi})$
$F_6(0)$	0.0	
$F_7(0)$	0.000083957	$(= 1/6720\sqrt{\pi})$

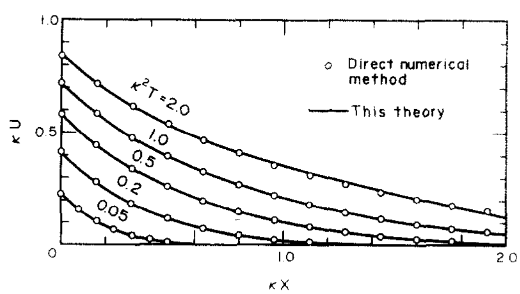

FIG. 2. Transient distribution of the dimensionless concentration for the case of $\mathbf{D}(U) = (1 - \kappa U)^2$.

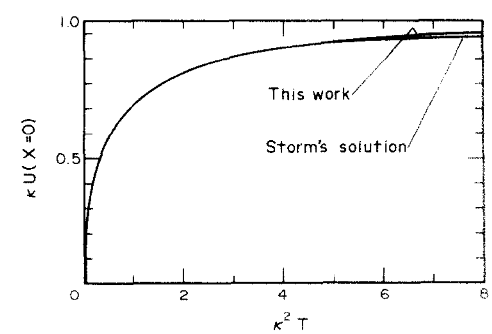

FIG. 3. Transient change of the concentration at the surface [for the case of $\mathbf{D}(U) = (1 - \kappa U)^2$].

Storm's transformations;

$$\left.\begin{aligned} U_s &= \int_{\phi}^{\phi_i} \sqrt{D} \, \mathrm{d}\phi \\ z &= \int_0^x \frac{1}{\sqrt{D}} \mathrm{d}x \\ \zeta &= \exp(A \cdot U_s) \end{aligned}\right\} \tag{42}$$

reduce the fundamental equation (3) to the following linear partial differential equation [3].

$$\left.\begin{aligned} &\frac{\partial \zeta}{\partial t} - AN_i \frac{\partial \zeta}{\partial z} = \frac{\partial^2 \zeta}{\partial z^2} \\ &\zeta = 1 \qquad \text{at} \quad t = 0 \\ &\frac{\partial \zeta}{\partial z} = -AN_i \quad \text{at} \quad z = 0. \end{aligned}\right\} \tag{43}$$

The exact solution is given by;

$$\zeta = 1 + \frac{\exp(\chi)}{2}(\nu^2 + 1 + \chi)\operatorname{erfc}\left(\frac{\chi}{2\nu} + \frac{\nu}{2}\right) - \tfrac{1}{2}\operatorname{erfc}\left(\frac{\chi}{2\nu} - \frac{\nu}{2}\right) - \frac{\nu}{\sqrt{\pi}}\exp\left\{-\left(\frac{\chi}{2\nu} - \frac{\nu}{2}\right)^2\right\} \tag{44}$$

where the auxiliary variables ν and χ are defined by;

$$\left.\begin{aligned} \nu &= AN_i\sqrt{t} \\ \chi &= -AN_i z. \end{aligned}\right\} \tag{45}$$

The transient change of U at the surface ($X = 0$) calculated from this strict solution (44) is also shown in Fig. 3.

The unsteady state distribution of $U(T, X)$, however, can not be evaluated from this solution, because equation (44) is not explicit form in the coordinate X. Direct numerical calculation of the finite difference equation for the original equation (6), then, were preformed. The numerical results are also shown in Fig. 2. Fairly good agreement can be seen from these figures especially when the dimensionless time $\kappa^2 T$ is less than unity.

2. *Exponential diffusion coefficient*

If the diffusion coefficient is represented by an exponential function of the concentration;

$$D(\phi) = A \cdot \exp(B \cdot \phi) \tag{46}$$

the Kirchhoff transformation yields

$$\left.\begin{aligned} \mathbf{D}(U) &= 1 - \kappa \cdot U \\ \kappa &= \mathbf{N}_i \cdot \ln E. \end{aligned}\right\} \tag{47}$$

The coefficients a_j in equation (12) then become;

$$\left.\begin{aligned} &a_1 = 1 \\ &a_2 = a_3 = \ldots = 0. \end{aligned}\right\} \tag{48}$$

The solutions for the ordinary differential equations (29) then become;

$$\left.\begin{aligned} F_1 &= 2\mathbf{E}_1 \\ F_2 &= -3\mathbf{E}_2 + 2\mathbf{E}_2\mathbf{E}_0. \end{aligned}\right\} \tag{49}$$

These analytical solutions and the further numerical solutions are shown in Fig. 4 and listed in Table 3.

The transient change of the surface concentration $U(X = 0)$ calculated from these values is shown in Fig. 5. Good agreement with the direct numerical solution can be seen from this figure.

3. *The other examples*

If the coefficients a_j for the power series of the relative diffusivity (12) can be given, the simultaneous linear ordinary differential equation (29) can be solved by analytical or numerical method. Though the general solutions for these equations can be easily obtained in the theoretical treatment, the singular solutions for each equation can be hardly found out especially when the higher terms of the coefficients are not eliminated.

The numerical method such as the shooting method with the Runge–Kutta's algorithm is available in this case. The numerical results for various cases of the diffusivity are listed in Table 4 and the transient changes of the surface concentration $U(X = 0)$ calculated from these values are shown in Fig. 6.

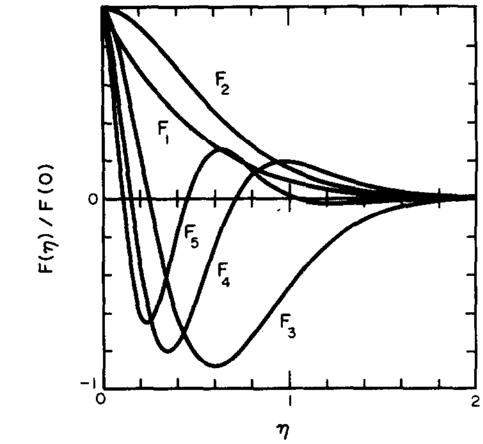

FIG. 4. Examples of the functions F_j for the exponential diffusivity case.

Table 3. Values of $F_j(0)$ for $D = \exp(B\phi)$

$F_1(0)$	1.128379167	$(= 2/\sqrt{\pi})$
$F_2(0)$	−0.25	$(= -1/4)$
$F_3(0)$	−0.016840651	
$F_4(0)$	−0.004261364	
$F_5(0)$	−0.001501915	
$F_6(0)$	−0.000624417	

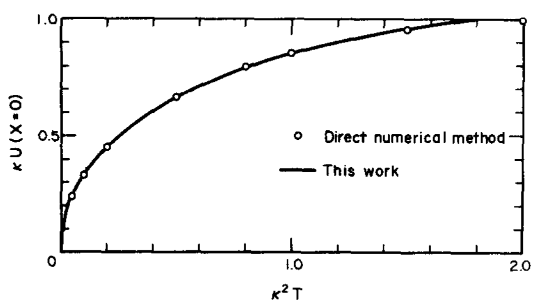

FIG. 5. Transient change of the surface concentration for the case of exponential diffusivity.

Table 4. Examples of $F_j(0)$

$\mathbf{D}(U)$	$(1-\kappa U)^{1/2}$	$(1-\kappa U)^{3/2}$	$(1-\kappa U)^3$
$F_1(0)$	1.128379167	1.128379167	1.128379167
$F_2(0)$	−0.125	−0.375	−0.75
$F_3(0)$	−0.024384438	0.022631360	0.332616746
$F_4(0)$	−0.009226964	0.002834917	−0.083710965
$F_5(0)$	−0.004522333	0.000297401	−0.006208076
$F_6(0)$	−0.002554149	0.000039061	0.016296072

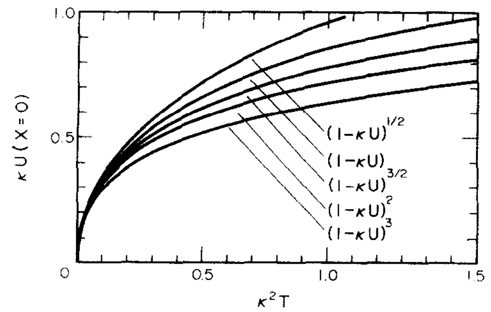

FIG. 6. Examples of the transient change of the surface concentrations.

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APPENDIX

We are concerned with the function $\mathbf{E}_n(\eta)$ which is defined by the following recurrence relations;

$$\left.\begin{aligned} \mathbf{E}_n(\eta) &= -\frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{E}_{n+1}(\eta), \quad n = -1, 0, 1, 2, \ldots \\ \mathbf{E}_{-1}(\eta) &= \frac{2}{\sqrt{\pi}}\exp(-\eta^2). \end{aligned}\right\} \tag{A.1}$$

This function coincides with the repeated integral of the error function when n is positive integer or zero,

$$\mathbf{E}_n(\eta) = i^n \operatorname{erfc}(\eta). \tag{A.2}$$

If these relations are extended to negative n, $\mathbf{E}_n$'s are represented by the following exponential functions;

$$\left.\begin{aligned} \mathbf{E}_{-1}(\eta) &= \frac{2}{\sqrt{\pi}}\exp(-\eta^2) \\ \mathbf{E}_{-2}(\eta) &= \frac{4}{\sqrt{\pi}}\eta\cdot\exp(-\eta^2) \\ \mathbf{E}_{-3}(\eta) &= -\frac{4}{\sqrt{\pi}}\exp(-\eta^2)+\frac{8}{\sqrt{\pi}}\eta^2\cdot\exp(-\eta^2) \\ &\ldots \\ &\ldots. \end{aligned}\right\} \tag{A.3}$$

It can be easily proved that

$$2n\mathbf{E}_n = -2\eta\mathbf{E}_{n-1}+\mathbf{E}_{n-2} \tag{A.4}$$

and that

$$\mathbf{E}_n''+2\eta\mathbf{E}_n'-2n\mathbf{E}_n = 0. \tag{A.5}$$

From these relations, it can also be derived that

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(\mathbf{E}_p\mathbf{E}_q)+2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}(\mathbf{E}_p\mathbf{E}_q)-2(p+q)\mathbf{E}_p\mathbf{E}_q = 2\mathbf{E}_{p-1}\mathbf{E}_{q-1} \tag{A.6}$$

and that

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(\mathbf{E}_p\mathbf{E}_q\mathbf{E}_r)+2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}(\mathbf{E}_p\mathbf{E}_q\mathbf{E}_r)-2(p+q+r)\mathbf{E}_p\mathbf{E}_q\mathbf{E}_r \\ &\quad = 2(\mathbf{E}_{p-1}\mathbf{E}_{q-1}\mathbf{E}_r+\mathbf{E}_{p-1}\mathbf{E}_q\mathbf{E}_{r-1}+\mathbf{E}_p\mathbf{E}_{q-1}\mathbf{E}_{r-1}). \end{aligned} \tag{A.7}$$

The general solution for equation (37) is then described by:

$$F_1 = A\cdot\mathbf{E}_1(\eta)+B\cdot\mathbf{E}_1(-\eta) \tag{A.8}$$

where A and B are constants.

Singular solution for equation (37) is zero. The boundary conditions then reduce equation (A.8) to the particular solution:

$$F_1 = 2\mathbf{E}_1. \tag{A.9}$$

Then equation (38) becomes

$$F_2''+2\eta F_2'-4F_2 = 8\mathbf{E}_1\mathbf{E}_{-1}. \tag{A.10}$$

The general solution for this equation is represented by

$$A\cdot\mathbf{E}_2(\eta)+B\cdot\mathbf{E}_2(-\eta). \tag{A.11}$$

Equation (A.6) shows that the singular solution of equation (A.10) is represented by

$$4\mathbf{E}_2\mathbf{E}_0. \tag{A.12}$$

Then, the particular solution satisfying the boundary conditions becomes;

$$F_2 = -6\mathbf{E}_2+4\mathbf{E}_2\mathbf{E}_0. \tag{A.13}$$

Equation (39) then becomes

$$\begin{aligned} &F_3''+2\eta F_3'-6F_3 \\ &\quad = -24(\mathbf{E}_2\mathbf{E}_{-1}+\mathbf{E}_1\mathbf{E}_0)+8(\mathbf{E}_1\mathbf{E}_1\mathbf{E}_{-1}-2\mathbf{E}_2\mathbf{E}_1\mathbf{E}_{-2}) \\ &\qquad +16(\mathbf{E}_1\mathbf{E}_1\mathbf{E}_{-1}+\mathbf{E}_1\mathbf{E}_0\mathbf{E}_0+\mathbf{E}_2\mathbf{E}_0\mathbf{E}_{-1}). \end{aligned} \tag{A.14}$$

The general and singular solutions are

$$A\cdot\mathbf{E}_3(\eta)+B\cdot\mathbf{E}_3(-\eta) \tag{A.15}$$

and

$$-12(\mathbf{E}_3\mathbf{E}_0+\mathbf{E}_2\mathbf{E}_1)+4\mathbf{E}_2\mathbf{E}_2\mathbf{E}_1+8\mathbf{E}_2\mathbf{E}_1\mathbf{E}_0. \tag{A.16}$$

The particular solution which satisfies the boundary conditions is

$$\begin{aligned} F_3 = 16\mathbf{E}_3 &- 12(\mathbf{E}_3\mathbf{E}_0+\mathbf{E}_2\mathbf{E}_1) \\ &+4\mathbf{E}_2\mathbf{E}_2\mathbf{E}_{-1}+8\mathbf{E}_2\mathbf{E}_1\mathbf{E}_0. \end{aligned} \tag{A.17}$$

If we repeat the same procedure, we may obtain the further analytical solutions.

NOUVELLE METHODE ANALYTIQUE DE RESOLUTION D'UN PROBLEME NON LINEAIRE DE DIFFUSION

Résumé—Une équation parabolique et non linéaire aux dérivées partielles, avec une diffusivité dépendant de la concentration est résolue par une méthode analytique dans un milieu semi-infini avec une condition aux limites de flux constant. L'équation est transformée en un système d'équations différentielles ordinaires linéaires. Les solutions sont représentées par une série de produits d'intégrales répétées de la fonction d'erreur.

EINE NEUE ANALYTISCHE METHODE ZUR LÖSUNG EINES NICHTLINEAREN DIFFUSIONSPROBLEMS

Zusammenfassung—Die nichtlineare, parabolische partielle Differentialgleichung mit konzentrationsabhängigem Diffusionskoeffizienten wird mit Hilfe einer analytischen Methode für einen halbunendlichen Bereich mit konstantem Massenstrom als Randbedingung gelöst. Dabei wird die Differentialgleichung in ein System gewöhnlicher, linearer Differentialgleichungen transformiert. Die Lösungen werden durch eine Produktentwicklung aus wiederholten Integralen der Fehlerfunktion dargestellt.

НОВЫЙ АНАЛИТИЧЕСКИЙ МЕТОД РЕШЕНИЯ НЕЛИНЕЙНОЙ ЗАДАЧИ ДИФФУЗИИ

Аннотация — Дается аналитическое решение нелинейного параболического дифференциального уравнения в частных производных с зависящим от концентрации коэффициентом диффузии для полубесконечной области при постоянной величине потока на границе. Рассматриваемое дифференциальное уравнение преобразовывается в систему обыкновенных линейных взаимосвязанных дифференциальных уравнений. Решения представлены в виде рядов, состоящих из произведений многократных интегралов от функции ошибок.